

Anisotropic Hydraulic Permeability Under Finite Deformation

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The structural organization of biological tissues and cells often produces anisotropic transport properties. These tissues may also undergo large deformations under normal function, potentially inducing further anisotropy. A general framework for formulating constitutive relations for anisotropic transport properties under finite deformation is lacking in the literature. This study presents an approach based on representation theorems for symmetric tensor-valued functions and provides conditions to enforce positive semidefiniteness of the permeability or diffusivity tensor. Formulations are presented, which describe materials that are orthotropic, transversely isotropic, or isotropic in the reference state, and where large strains induce greater anisotropy. Strain-induced anisotropy of the permeability of a solid-fluid mixture is illustrated for finite torsion of a cylinder subjected to axial permeation. It is shown that, in general, torsion can produce a helical flow pattern, rather than the rectilinear pattern observed when adopting a more specialized, unconditionally isotropic spatial permeability tensor commonly used in biomechanics. The general formulation presented in this study can produce both affine and nonaffine reorientations of the preferred directions of material symmetry with strain, depending on the choice of material functions. This study addresses a need in the biomechanics literature by providing guidelines and formulations for anisotropic strain-dependent transport properties in porous-deformable media undergoing large deformations. [DOI: 10.1115/1.4002588]

1 Introduction

Biological soft tissues can be described as mixtures of a solid matrix and an interstitial fluid. If the solid matrix exhibits structural anisotropy, the transport properties of the fluid through the solid matrix may also be anisotropic [1,2]. For example, for a collagenous tissue that exhibits preferred fiber orientations, permeability of the solvent or diffusivity of the solutes in the directions parallel and perpendicular to the fibers may have different values [3–10]. Therefore, in general, transport properties are formulated in tensorial form, such as the permeability or diffusivity tensors.

When tissues undergo finite deformation, the material symmetry in the deformed configuration may be different than the material symmetry in the reference configuration. A material that exhibits isotropic transport properties in the reference configuration may develop strain-induced anisotropy under finite deformation. To the best of our knowledge, a general approach for formulating anisotropic properties for the permeability tensor while accounting for the possibility of strain-induced anisotropy under finite deformation has not been presented previously. Such a formulation must take into account that the permeability tensor is symmetric and positive semidefinite. This study presents a general approach based on representation theorems for symmetric tensor-valued functions [11,12].

2 Formulation

2.1 Permeability Tensor. The permeability tensor describes frictional interactions between a porous solid matrix and its interstitial fluid. From the equation of conservation of linear momentum for the fluid, when the effects of viscosity are neglected rela-

tive to the effects of permeability, under quasistatic conditions and in the absence of external body forces, the basic relation for the flux of interstitial fluid relative to the solid is

$$\mathbf{w} = -\mathbf{k} \cdot \text{grad } p \quad (1)$$

where \mathbf{w} is the relative fluid flux, \mathbf{k} is the second-order hydraulic permeability tensor, and p is the interstitial fluid pressure. Based on the second law of thermodynamics, \mathbf{k} must be symmetric and positive semidefinite (see Appendix).

2.2 Material and Spatial Frames. The relation of Eq. (1) is given in the spatial configuration; therefore, \mathbf{k} represents the spatial permeability tensor. The corresponding permeability tensor in the material frame is obtained using the standard Piola transformation [13]:

$$\mathbf{K} = J\mathbf{F}^{-1} \cdot \mathbf{k} \cdot \mathbf{F}^{-T}, \quad (2)$$

where \mathbf{F} is the deformation gradient for the solid matrix and $J = \det \mathbf{F}$. Equivalently,

$$\mathbf{k} = J^{-1}\mathbf{F} \cdot \mathbf{K} \cdot \mathbf{F}^T \quad (3)$$

recovers the permeability tensor in the spatial frame [14].

In a nonlinear finite element implementation of mixture analysis under finite strain, it is necessary to evaluate the linearization of the permeability tensor along increments $\Delta \mathbf{u}$ in the solid matrix displacement:

$$D\mathbf{K}[\Delta \mathbf{u}] = \mathbf{K} : \frac{1}{2} D\mathbf{C}[\Delta \mathbf{u}] \quad (4)$$

where $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ is the right Cauchy–Green tensor, D is the directional derivative operator [13],¹ and

¹ $Df(\mathbf{x})[\mathbf{u}] = d/d\varepsilon|_{\varepsilon=0} f(\mathbf{x} + \varepsilon \mathbf{u})$ for any function $f(\mathbf{x})$.

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$$\mathbf{K} = 2 \frac{\partial \mathbf{K}}{\partial \mathbf{C}} \quad (5)$$

is a fourth-order tensor representing the rate of change of \mathbf{K} with strain in the material frame. The corresponding tensor in the spatial frame is obtained using the Piola transformation for fourth-order tensors, using the tensor product defined by Curnier et al. [15].²

$$\mathbf{k} = J^{-1}(\mathbf{F} \otimes \mathbf{F}) : \mathbf{K} : (\mathbf{F}^T \otimes \mathbf{F}^T) \quad (6)$$

Based on the relations of Eqs. (5) and (6) and the symmetry of \mathbf{K} and \mathbf{C} , \mathbf{k} and \mathbf{K} exhibit minor symmetries (for example, $k_{ijkl} = k_{jikl}$ and $k_{ijkl} = k_{ijlk}$) but not necessarily a major symmetry ($\mathbf{k} \neq \mathbf{k}^T$ or equivalently, $k_{ijkl} \neq k_{klij}$).

2.3 Constraints on Constitutive Relations. The dependence of the permeability tensor on the deformation of the porous solid matrix must be specified via a constitutive relation. Except for the requirements of symmetry and positive semidefiniteness, there are no other formal thermodynamic constraints on this formulation. However, the permeability should reduce to zero in the limit of pore closure, in order to properly produce zero fluid flux according to Eq. (1). Let the solid and fluid volume fractions be given by φ^s and φ^f , respectively. The mixture saturation condition (the absence of voids) may be represented by

$$\varphi^s + \varphi^f = 1 \quad (7)$$

When each constituent is intrinsically incompressible, the conservation of mass for the porous solid matrix implies that

$$\varphi^s = \varphi_r^s J \quad (8)$$

where φ_r^s is the solid volume fraction in the reference state ($\mathbf{C} = \mathbf{I}$).

In the analysis of saturated solid-fluid mixtures, pore closure occurs when all the fluid has been squeezed out of the pores, $\varphi^f = 0$ and $\varphi^s = 1$, implying that $J = \varphi_r^s$ at pore closure. Therefore, an additional constraint that may be placed on constitutive relations for the permeability tensor is the requirement that it reduces to zero in the limit of pore closure,

$$\lim_{J \rightarrow \varphi_r^s} \mathbf{K} = \mathbf{0}, \quad \lim_{J \rightarrow \varphi_r^s} \mathbf{k} = \mathbf{0} \quad (9)$$

2.4 Representation Theorem for Tensor-Valued Functions.

This study seeks to formulate a framework for constitutive modeling of the permeability tensor, which is a second-order symmetric tensor function of the strain. A general approach is sought, which can subsequently be specialized to account for various constraints specific to the permeability tensor. Therefore, the formulation starts from basic principles embodied in representation theorems for symmetric tensor-valued functions [11,12]. These representation theorems can be formulated for various postulated material symmetries, although the literature generally describes representations for orthotropic and higher symmetries. Therefore, the presentation starts with materials whose transport properties are orthotropic, and then proceeds to higher symmetries, including isotropy and transverse isotropy.

2.4.1 Orthotropy. An orthotropic material has three orthogonal planes of symmetry. Let the vectors \mathbf{A}_a represent the unit normal vectors to these planes in the material frame ($a=1,2,3$). Then, according to the representation theorem for orthotropic symmetric tensor-valued functions [11,12,16], the functional dependence of the permeability tensor in the material frame is given by $\mathbf{K}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{C})$, where $\mathbf{A}_a \cdot \mathbf{A}_b = \delta_{ab}$ (Kronecker delta), and the most general dependence on strain is given by

$$\mathbf{K} = K_0 \mathbf{C}^{-1} + \sum_{a=1}^3 K_1^a \mathbf{M}_a + K_2^a (\mathbf{M}_a \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{M}_a), \quad \mathbf{M}_a = \mathbf{A}_a \otimes \mathbf{A}_a \quad (10)$$

where the scalars K_0 , K_1^a , and K_2^a are functions of the following strain invariants:

$$I_3 = \det \mathbf{C}, \quad I_{a+3} = \mathbf{M}_a : \mathbf{C}, \quad I_{a+6} = \mathbf{M}_a : \mathbf{C}^2, \quad a = 1, 2, 3 \quad (11)$$

\mathbf{C} and \mathbf{C}^{-1} are positive definite for any arbitrary state of strain (their eigenvalues are always positive); based on its definition, \mathbf{M}_a is similarly a positive semidefinite tensor. Since the sum of positive definite tensors produces a positive definite tensor [17], to ensure that \mathbf{K} is positive semidefinite for any arbitrary state of strain, it is necessary and sufficient to formulate the scalar functions K_i such that

$$K_i(I_j) \geq 0, \quad \forall I_j, j = 3, \dots, 9 \quad (12)$$

In this expression, K_i is used generically to represent K_0 , K_1^a , and K_2^a . These constraints are thus specific to the constraint of positive semidefiniteness for the permeability tensor. In addition, to satisfy the constraint of Eq. (9), it is necessary and sufficient to satisfy

$$\lim_{J \rightarrow \varphi_r^s} K_i = 0 \quad (13)$$

where $J = \sqrt{I_3}$.

Using Eq. (3), the permeability tensor in the spatial frame is given by

$$\mathbf{k} = k_0 \mathbf{I} + \sum_{a=1}^3 k_1^a \mathbf{m}_a + k_2^a (\mathbf{m}_a \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_a), \quad \mathbf{m}_a = \mathbf{F} \cdot \mathbf{M}_a \cdot \mathbf{F}^T \quad (14)$$

where

$$k_0 = J^{-1} K_0, \quad k_1^a = J^{-1} K_1^a, \quad k_2^a = J^{-1} K_2^a \quad (15)$$

and $\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T$ is the left Cauchy-Green tensor.

The derivative of \mathbf{K} with respect to the strain may be obtained from Eq. (5) using the chain rule of differentiation,³

$$\mathbf{K} = \mathbf{C}^{-1} \otimes \hat{\mathbf{K}}_0 - 2K_0 \mathbf{C}^{-1} \bar{\otimes} \mathbf{C}^{-1} + \sum_{a=1}^3 \mathbf{M}_a \otimes \hat{\mathbf{K}}_1^a + \sum_{a=1}^3 (\mathbf{M}_a \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{M}_a) \otimes \hat{\mathbf{K}}_2^a + 2K_2^a (\mathbf{M}_a \bar{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{M}_a) \quad (16)$$

where

$$\hat{\mathbf{K}}_i = 2 \frac{\partial K_i}{\partial \mathbf{C}} = 2 \left[I_3 \frac{\partial K_i}{\partial I_3} \mathbf{C}^{-1} + \sum_{b=1}^3 \frac{\partial K_i}{\partial I_{b+3}} \mathbf{M}_b + \frac{\partial K_i}{\partial I_{b+6}} (\mathbf{M}_b \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{M}_b) \right] \quad (17)$$

In this expression, $\hat{\mathbf{K}}_i$ is used generically to represent $\hat{\mathbf{K}}_0$, $\hat{\mathbf{K}}_1^a$, and $\hat{\mathbf{K}}_2^a$. Using Eq. (6), the spatial representation of this fourth-order tensor becomes

$$\mathbf{k} = \mathbf{I} \otimes \hat{\mathbf{k}}_0 - 2k_0 \mathbf{I} \bar{\otimes} \mathbf{I} + \sum_{a=1}^3 \mathbf{m}_a \otimes \hat{\mathbf{k}}_1^a + \sum_{a=1}^3 (\mathbf{m}_a \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_a) \otimes \hat{\mathbf{k}}_2^a + 2k_2^a (\mathbf{m}_a \bar{\otimes} \mathbf{b} + \mathbf{b} \bar{\otimes} \mathbf{m}_a) \quad (18)$$

where

³The tensor products \otimes and $\bar{\otimes}$ between second-order tensors \mathbf{A} and \mathbf{B} are, respectively, defined by $(\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ij} B_{kl}$ and $(\mathbf{A} \bar{\otimes} \mathbf{B})_{ijkl} = (A_{ik} B_{jl} + A_{il} B_{jk})/2$ in a Cartesian basis [15].

²The tensor product \otimes between second-order tensors \mathbf{A} and \mathbf{B} is defined by $(\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ik} B_{jl}$ in a Cartesian basis [15].

$$\hat{\mathbf{k}}_i = J^{-1} \mathbf{F} \cdot \hat{\mathbf{K}}_i \cdot \mathbf{F}^T = 2J^{-1} \left[I_3 \frac{\partial K_i}{\partial I_3} \mathbf{I} + \sum_{b=1}^3 \frac{\partial K_i}{\partial I_{b+3}} \mathbf{m}_b + \frac{\partial K_i}{\partial I_{b+6}} (\mathbf{m}_b \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_b) \right] \quad (19)$$

$\hat{\mathbf{k}}_i$ is used generically to represent $\hat{\mathbf{k}}_0$, $\hat{\mathbf{k}}_1^a$, and $\hat{\mathbf{k}}_2^a$.

Due to the symmetry of \mathbf{I} , \mathbf{C} , \mathbf{C}^{-1} , and \mathbf{M}_a , it follows from Eq. (16) that \mathbf{K} satisfies the two minor symmetries, but not the major symmetry ($\mathbf{K}^T \neq \mathbf{K}$ in general); the same argument applies to \mathbf{k} .

2.4.2 Isotropy. The case of isotropy may be derived from the representation theorem for isotropic tensor-valued functions, or simply reduced from the orthotropic case presented above. For isotropic materials, there is no preferred material direction, so that

$$K_1^a \equiv K_1, \quad K_2^a \equiv K_2, \quad a = 1, 2, 3 \quad (20)$$

In this case, since $\sum_{a=1}^3 \mathbf{M}_a = \mathbf{I}$ and

$$I_1 = \mathbf{I} : \mathbf{C} = \sum_{a=1}^3 I_{a+3}, \quad I_2 = \mathbf{I} : \mathbf{C}^2 = \sum_{a=1}^3 I_{a+6} \quad (21)$$

the permeability tensors in the material and spatial frames reduce to

$$\mathbf{K} = K_0 \mathbf{C}^{-1} + K_1 \mathbf{I} + 2K_2 \mathbf{C} \quad (22)$$

$$\mathbf{k} = k_0 \mathbf{I} + k_1 \mathbf{b} + 2k_2 \mathbf{b}^2 \quad (23)$$

where all K_i s depend on I_1, I_2, I_3 . The derivative of the material permeability tensor with respect to the strain becomes

$$\mathbf{K} = \mathbf{C}^{-1} \otimes \hat{\mathbf{K}}_0 - 2K_0 \mathbf{C}^{-1} \otimes \bar{\mathbf{I}} \mathbf{C}^{-1} + \mathbf{I} \otimes \hat{\mathbf{K}}_1 + 2\mathbf{C} \otimes \hat{\mathbf{K}}_2 + 4K_2 \mathbf{I} \otimes \bar{\mathbf{I}} \quad (24)$$

where

$$\hat{\mathbf{K}}_i = 2 \left(I_3 \frac{\partial K_i}{\partial I_3} \mathbf{C}^{-1} + \frac{\partial K_i}{\partial I_1} \mathbf{I} + 2 \frac{\partial K_i}{\partial I_2} \mathbf{C} \right) \quad (25)$$

Similarly, for the spatial representation,

$$\mathbf{k} = \mathbf{I} \otimes \hat{\mathbf{k}}_0 - 2k_0 \mathbf{I} \otimes \bar{\mathbf{I}} \mathbf{I} + \mathbf{b} \otimes \hat{\mathbf{k}}_1 + 2\mathbf{b}^2 \otimes \hat{\mathbf{k}}_2 + 4k_2 \mathbf{b} \otimes \bar{\mathbf{b}} \quad (26)$$

where

$$\hat{\mathbf{k}}_i = 2J^{-1} \left(I_3 \frac{\partial K_i}{\partial I_3} \mathbf{I} + \frac{\partial K_i}{\partial I_1} \mathbf{b} + 2 \frac{\partial K_i}{\partial I_2} \mathbf{b}^2 \right) \quad (27)$$

2.4.3 Transverse Isotropy. The case of transverse isotropy may be similarly reduced from orthotropy by letting

$$K_j^1 = K_j^2 \equiv K_j^T, \quad K_j^3 \equiv K_j^A, \quad j = 1, 2 \quad (28)$$

where K_j^T and K_j^A represent permeability functions in the transverse plane of isotropy and along the axial direction normal to that plane, respectively. Then, the permeability tensors are given by

$$\mathbf{K} = K_0 \mathbf{C}^{-1} + K_1^T \mathbf{I} + 2K_2^T \mathbf{C} + (K_1^A - K_1^T) \mathbf{M}_3 + (K_2^A - K_2^T) (\mathbf{M}_3 \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{M}_3) \quad (29)$$

$$\mathbf{k} = k_0 \mathbf{I} + k_1^T \mathbf{b} + 2k_2^T \mathbf{b}^2 + (k_1^A - k_1^T) \mathbf{m}_3 + (k_2^A - k_2^T) (\mathbf{m}_3 \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_3) \quad (30)$$

where all K_i s are functions of I_1, I_2, I_3, I_6, I_9 .⁴ Similarly,

⁴Conventionally, in a transversely isotropic formulation, the strain invariants $\mathbf{M}_3 : \mathbf{C}$ and $\mathbf{M}_3 : \mathbf{C}^2$ are denoted by I_4 and I_5 , respectively. They appear as I_6 and I_9 in this treatment to maintain a consistent notation between orthotropic and transversely isotropic formulations.

$$\begin{aligned} \mathbf{K} = & \mathbf{C}^{-1} \otimes \hat{\mathbf{K}}_0 - 2K_0 \mathbf{C}^{-1} \otimes \bar{\mathbf{I}} \mathbf{C}^{-1} + \mathbf{I} \otimes \hat{\mathbf{K}}_1^T + 2\mathbf{C} \otimes \hat{\mathbf{K}}_2^T + 4K_2^T \mathbf{I} \otimes \bar{\mathbf{I}} \\ & + \mathbf{M}_3 \otimes (\hat{\mathbf{K}}_1^A - \hat{\mathbf{K}}_1^T) + (\mathbf{M}_3 \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{M}_3) \otimes (\hat{\mathbf{K}}_2^A - \hat{\mathbf{K}}_2^T) + 2(K_2^A \\ & - K_2^T) (\mathbf{M}_3 \otimes \bar{\mathbf{I}} \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{I}} \mathbf{M}_3) \end{aligned} \quad (31)$$

where

$$\hat{\mathbf{K}}_i = 2 \left(I_3 \frac{\partial K_i}{\partial I_3} \mathbf{C}^{-1} + \frac{\partial K_i}{\partial I_1} \mathbf{I} + 2 \frac{\partial K_i}{\partial I_2} \mathbf{C} + \frac{\partial K_i}{\partial I_6} \mathbf{M}_3 + \frac{\partial K_i}{\partial I_9} (\mathbf{M}_3 \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{M}_3) \right) \quad (32)$$

and

$$\begin{aligned} \mathbf{k} = & \mathbf{I} \otimes \hat{\mathbf{k}}_0 - 2k_0 \mathbf{I} \otimes \bar{\mathbf{I}} \mathbf{I} + \mathbf{b} \otimes \hat{\mathbf{k}}_1^T + 2\mathbf{b}^2 \otimes \hat{\mathbf{k}}_2^T + 4k_2^T \mathbf{b} \otimes \bar{\mathbf{b}} + \mathbf{m}_3 \otimes (\hat{\mathbf{k}}_1^A \\ & - \hat{\mathbf{k}}_1^T) + (\mathbf{m}_3 \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_3) \otimes (\hat{\mathbf{k}}_2^A - \hat{\mathbf{k}}_2^T) + 2(k_2^A - k_2^T) (\mathbf{m}_3 \otimes \bar{\mathbf{b}} + \bar{\mathbf{b}} \\ & \otimes \mathbf{m}_3) \end{aligned} \quad (33)$$

where

$$\hat{\mathbf{k}}_i = 2J^{-1} \left(I_3 \frac{\partial K_i}{\partial I_3} \mathbf{I} + \frac{\partial K_i}{\partial I_1} \mathbf{b} + 2 \frac{\partial K_i}{\partial I_2} \mathbf{b}^2 + \frac{\partial K_i}{\partial I_6} \mathbf{m}_3 + \frac{\partial K_i}{\partial I_9} (\mathbf{m}_3 \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_3) \right) \quad (34)$$

2.5 Examples of Constitutive Relations. The most common constitutive relation for the hydraulic permeability tensor adopted in the biomechanics literature has the form

$$\mathbf{k} = k_0(J) \mathbf{I} \quad (35)$$

where the specific dependence of k_0 on J is often an exponential or power-law function [18,19], such as

$$k_0(J) = k_{0r} \left(\frac{J - \varphi_r^s}{1 - \varphi_r^s} \right)^{\kappa_0} e^{M_0(J^2 - 1)/2} \quad (36)$$

where k_{0r} , κ_0 and M_0 are material constants to be determined from experiments [18]. This function satisfies the constraints of Eqs. (12) and (13). This formulation represents a special case of permeability as given in Eq. (14), with $k_1^a = k_2^a = 0$, which does not exhibit strain-induced anisotropy. Regardless of the state of strain in the material, \mathbf{k} remains an isotropic tensor,⁵ implying that every plane through the material represents a plane of material symmetry for transport properties. It may be noted that this constitutive relation, formulated in the spatial frame, corresponds to $\mathbf{K} = K_0 \mathbf{C}^{-1}$ in the material frame, where $K_0 = Jk_0$ as per Eq. (15). Thus, in the material frame, \mathbf{K} is not an isotropic tensor for arbitrary states of strain.

For this special case of constitutive relation, the corresponding spatial form of the derivative with respect to strain reduces to

$$\mathbf{k} = k_0 \left(\left[1 + J \left(\frac{\kappa_0}{J - \varphi_r^s} + M_0 J \right) \right] \mathbf{I} \otimes \mathbf{I} - 2\mathbf{I} \otimes \bar{\mathbf{I}} \mathbf{I} \right) \quad (37)$$

Anisotropic permeability has generally been employed in problems with small deformations, for which the spatial and material representations are approximately the same. The most common choice of constitutive relation has had the form $\mathbf{K} = K_1^T \mathbf{I} + (K_1^A - K_1^T) \mathbf{M}_3$ for transverse isotropy [10,20] or $\mathbf{K} = \sum_{a=1}^3 K_1^a \mathbf{M}_a$ for orthotropy [9,21]. These formulations represent special cases of transversely isotropic permeability as given in Eq. (29) and orthotropic permeability as given in Eq. (10), with $K_0 = K_2^a = 0$ and K_1^a treated as constant.

⁵A second-order tensor \mathbf{T} is isotropic when $\mathbf{T} = \alpha \mathbf{I}$ for any scalar α .

Constitutive relations are guided by experimental measurements. Since permeation experiments in deformable tissues are challenging to perform, most permeation studies under finite deformation have been performed in one-dimensional setups under unidirectional flow conditions [22,23]. More often, material coefficients for an assumed permeability constitutive relation under finite deformation were obtained by curve-fitting the load-deformation response of soft tissues in confined compression [24–28]. Therefore, because of the one-dimensional nature of these finite deformation experiments, there are limited experimental data in the literature that can shed insight into the potential mechanism of strain-induced anisotropy in the permeability. Yet, even in the case of an isotropic material, the more general formulation for the spatial permeability given in Eq. (23) suggests that large deformations may induce an anisotropic permeability tensor, when either $k_1 \neq 0$ or $k_2 \neq 0$.

It is possible that the choices of permeability constitutive relations adopted for biological tissues to date represent special cases that do not necessarily explore the full range of possible deformation-induced responses. To explore the implications of strain-induced anisotropy in the permeability, consider the general case of Eq. (23), which represents the response for a material whose permeability is isotropic in a strain-free state ($\mathbf{b}=\mathbf{I}$). Under general deformations ($\mathbf{b} \neq \mathbf{I}$), the permeability tensor is no longer an isotropic tensor. For arbitrary strain fields in an isotropic material, three planes of symmetry emerge, which correspond to the principal planes of normal strain (the planes whose normals are the eigenvectors of \mathbf{b}). Therefore, in a material whose transport properties are isotropic under a strain-free state, strain-induced anisotropy produces orthotropic symmetry with the orientation of the planes of symmetry varying with the state of strain.

If the permeability is anisotropic in a strain-free state, such as in the case of orthotropy (Eq. (14)), it becomes evident that this anisotropy is manifested in the terms other than $k_0\mathbf{I}$ appearing in these general relations. Therefore, anisotropic transport properties can only be specified with material functions $k_1^a \neq 0$ or $k_2^a \neq 0$. In a strain-free state, the three planes of symmetry are normal to \mathbf{A}_a ($a=1,2,3$) by definition. However, under a general state of strain (when the eigenvectors of \mathbf{b} do not align with $\mathbf{F} \cdot \mathbf{A}_a$), the strain-induced anisotropy degenerates to complete loss of symmetry (triclinic form). In effect, it is not possible to describe an orthotropic (or transversely isotropic or monoclinic) spatial permeability tensor \mathbf{k} whose planes of symmetry remain invariant with strain under general conditions.

For example, consider the special case of an orthotropic material, where $k_0=k_2^a=0$, so that

$$\mathbf{k} = \sum_{a=1}^3 k_1^a \mathbf{m}_a \quad (38)$$

It should be evident that the strain-induced anisotropy for this special case obeys an affine transformation of the preferred material directions \mathbf{A}_a , since the expression for \mathbf{m}_a in Eq. (14) may also be written as $\mathbf{m}_a = (\mathbf{F} \cdot \mathbf{A}_a) \otimes (\mathbf{F} \cdot \mathbf{A}_a)$. A constitutive form for k_1^a may be selected such that it produces the same permeability as the form of Eqs. (35) and (36) when considering the canonical problem of 1D permeation under finite deformation

$$k_1^a(J) = \frac{k_{1r}^a}{J^2} \left(\frac{J - \varphi_r^s}{1 - \varphi_r^s} \right)^{\kappa_1^a} e^{M_1^a(J^2-1)/2}, \quad a=1,2,3 \quad (39)$$

where k_{1r}^a , κ_1^a , and M_1^a represent material constants. The corresponding expression for \mathbf{k} becomes

$$\mathbf{k} = \sum_{a=1}^3 \left[J^2 M_1^a + \frac{J(\kappa_1^a - 1) + \varphi_r^s}{J - \varphi_r^s} \right] k_1^a \mathbf{m}_a \otimes \mathbf{I} \quad (40)$$

Another special case that may be considered here corresponds to $k_0=k_1^a=0$, such that

$$\mathbf{k} = \sum_{a=1}^3 k_2^a (\mathbf{m}_a \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_a) \quad (41)$$

For this choice, it is evident that the strain-induced anisotropy follows a nonaffine transformation of the preferred material directions \mathbf{A}_a , since $\mathbf{m}_a \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_a$ is neither equal nor proportional to \mathbf{m}_a . Once again, a constitutive form for k_2^a may be selected such that it produces the same permeability as the form of Eqs. (35) and (36) when considering the canonical problem of 1D permeation under finite deformation,

$$k_2^a(J) = \frac{k_{2r}^a}{2J^4} \left(\frac{J - \varphi_r^s}{1 - \varphi_r^s} \right)^{\kappa_2^a} e^{M_2^a(J^2-1)/2}, \quad a=1,2,3 \quad (42)$$

The corresponding expression for \mathbf{k} becomes

$$\mathbf{k} = \sum_{a=1}^3 k_2^a \left[\left(J^2 M_2^a + \frac{J(\kappa_2^a - 3) + 3\varphi_r^s}{J - \varphi_r^s} \right) (\mathbf{m}_a \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{m}_a) \otimes \mathbf{I} + 2(\mathbf{m}_a \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{m}_a) \right] \quad (43)$$

These constitutive relations can be reduced to transverse isotropy and isotropy as shown above. Clearly, any number of specialized forms and combinations may be adopted for these constitutive relations. As a final example, for a referentially isotropic permeability tensor, a choice of constitutive relation may be given by

$$\mathbf{k} = \left(k_0 \mathbf{I} + \frac{k_{1r}}{J^2} \mathbf{b} + \frac{k_{2r}}{J^4} \mathbf{b}^2 \right) \left(\frac{J - \varphi_r^s}{1 - \varphi_r^s} \right)^\kappa e^{M(J^2-1)/2} \quad (44)$$

In this formulation, prescribing various values to k_0 , k_{1r} , and k_{2r} can give preferential weight to affine or nonaffine strain-induced anisotropy. In the canonical problems analyzed below, some salient features of strain-induced anisotropy are explored.

3 Canonical Problems

3.1 Isochoric Axial Stretch. Consider the isochoric axial stretch of an isotropic homogeneous cylinder, in a cylindrical basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

$$[\mathbf{F}] = \begin{bmatrix} 1/\sqrt{\lambda} & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad (45)$$

for which $J=1$ (no change in pore volume with deformation). The matrix of the permeability tensor resulting from Eq. (23) is

$$[\mathbf{k}] = \begin{bmatrix} k_0 + \frac{k_1}{\lambda} + 2\frac{k_2}{\lambda^2} & 0 & 0 \\ 0 & k_0 + \frac{k_1}{\lambda} + 2\frac{k_2}{\lambda^2} & 0 \\ 0 & 0 & k_0 + k_1\lambda^2 + 2k_2\lambda^4 \end{bmatrix} \quad (46)$$

As λ increases above unity with axial stretching, the radial and circumferential permeability components, $k_0 + k_1/\lambda + 2k_2/\lambda^2$, will always be smaller than the axial component $k_0 + k_1\lambda^2 + 2k_2\lambda^4$, regardless of the functional dependence of k_i on I_j ($j=1,2,3$). Thus, even though the pore volume does not change, the shape of the pores is altered with deformation such that the permeability tensor is no longer isotropic (unless $k_1=k_2=0$); in this example, the symmetry reduces to transverse isotropy under axial stretching.

For example, consider that the isotropic permeability arises from a solid matrix consisting of randomly oriented fibers in the reference configuration, with each fiber orientation represented by

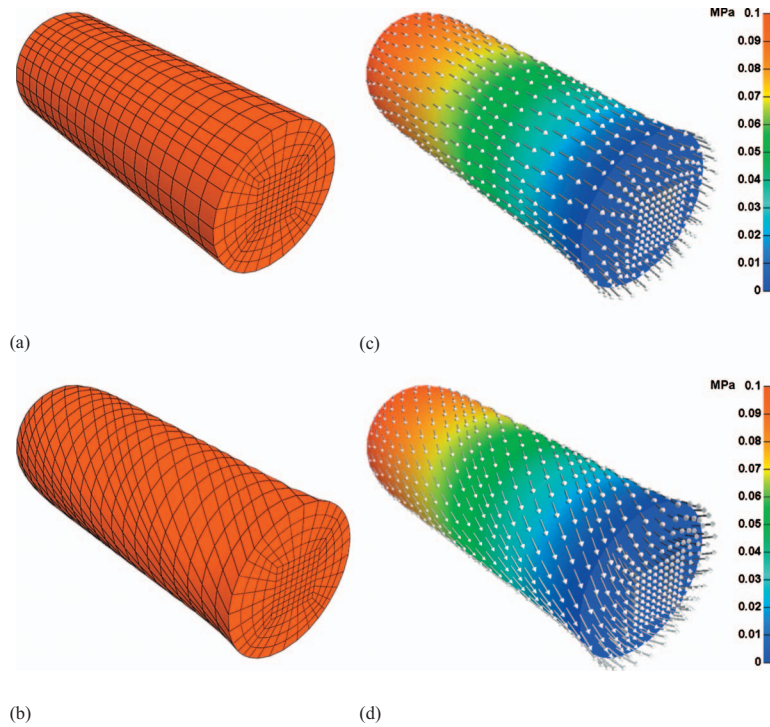


Fig. 1 Finite element results for permeation through a cylinder subjected to finite torsion: (a) cylinder geometry in reference configuration, (b) cylinder geometry after one half-turn (180 deg twist), (c) vector plot of relative volume flux of solvent in deformed configuration using $\mathbf{k}=\mathbf{k}_0\mathbf{I}$, which does not account for strain-induced anisotropy, and (d) vector plot of flux in deformed configuration using $\mathbf{k}=\mathbf{k}_1\mathbf{b}$, which accounts for strain-induced anisotropy

the vector $\mathbf{N}=\sin \phi \mathbf{e}_r+\cos \phi \mathbf{e}_z$, $0 \leq \phi < \pi$. Assuming that fibers deform according to an affine transformation, they become oriented along

$$\mathbf{n} = \frac{\mathbf{F} \cdot \mathbf{N}}{|\mathbf{F} \cdot \mathbf{N}|} = \frac{\sin \phi \mathbf{e}_r + \lambda^{3/2} \cos \phi \mathbf{e}_z}{\sqrt{\lambda^3 \cos^2 \phi + \sin^2 \phi}} \quad (47)$$

As the axial stretch λ increases, the fibers turn toward the long axis of the cylinder, eventually lining up with \mathbf{e}_z in the limit as $\lambda \rightarrow \infty$. Therefore, for this example of isochoric stretch of an isotropic material, according to Eq. (46), it is found that permeability becomes greater along the direction of preferred fiber reorientation than along the directions perpendicular to it.

The same conclusions would be reached for the case of confined axial stretch of a homogeneous isotropic cylinder when $[\mathbf{F}]=\text{diag}\{1, 1, \lambda\}$.

3.2 Finite Torsion. In this example, a finite element analysis was used to illustrate strain-induced anisotropy in a biphasic isotropic cylinder subjected to torsion, with a pressure gradient applied axially to promote permeation. The finite element implementation of these formulations was incorporated into FEBIO, an existing public-domain, open-source finite element program.⁶ The cylinder had a diameter of 10 mm and was 25 mm long. Its solid matrix was described as a neo-Hookean isotropic elastic solid, with Young's modulus of 1 MPa and Poisson's ratio set to zero. The permeability tensor was described by Eq. (44), where each of k_{0r} , k_{1r} , and k_{2r} was, respectively, set to $10^3 \text{ mm}^4/\text{N s}$, while the other two were set to zero. In all cases, the remaining material parameters were $\varphi_r^s=0.2$, $\kappa=2$, and $M=0$. One end of the cylinder was constrained to have zero displacements along all three coor-

ordinate directions and a prescribed fluid pressure of 0.1 MPa. The other end was prescribed to rotate in its plane by a half-rotation (180 deg twist) and constrained to zero displacement along the axial direction; the fluid pressure was also constrained to zero on this face (free-draining conditions). The lateral surface was traction-free and impermeable.

The steady-state response was examined here since the focus of this analysis was on the effect of strain-induced anisotropy on the relative fluid flux \mathbf{w} rather than the transient response to the applied boundary conditions.

Results showed that the streamlines of solvent volume flux remained rectilinear in the case of $\mathbf{k}=\mathbf{k}_0\mathbf{I}$ ($k_{0r} \neq 0$, $k_{1r}=k_{2r}=0$), regardless of the applied deformation (Fig. 1(c)). For the other two cases, however, the flux twisted into a helical pattern with torsion (Fig. 1(d)). A closer examination of the flux pattern, when superposed on the deformed finite element mesh, confirmed that the case $\mathbf{k}=\mathbf{k}_1\mathbf{b}$ produced an affine transformation, with the flux aligning exactly with the deformed mesh (Fig. 2(c)). In contrast, for the case $\mathbf{k}=2k_2\mathbf{b}^2$, the flux vector twisted further than the direction prescribed by an affine transformation (Fig. 2(d)).

4 Discussion

This study proposed a systematic approach for formulating constitutive relations for the anisotropic, strain-dependent hydraulic permeability tensor in deformable porous media. The hydraulic permeability is a material function that relates the flux of interstitial fluid to the pressure gradient, Eq. (1). A primary motivation for this aim was the observation that many biological tissues undergo large deformations during normal function, yet few guidelines are currently available in the literature for formulating strain-dependent anisotropic transport properties under such deformations.

⁶<http://mrl.sci.utah.edu/software?menu=Software>

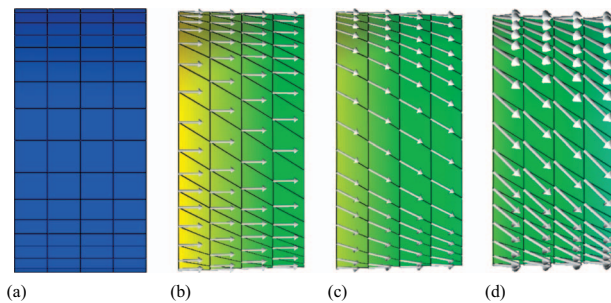


Fig. 2 Close-up of relative volume flux of solvent, in relation to mesh deformation: (a) reference configuration, (b) $k=k_0I$, (c) $k=k_1b$, and (d) $k=2k_2b^2$. Case (c) confirms that this constitutive relation produces strain-induced anisotropy that follows an affine transformation of preferred material directions since the flux realigns exactly with the deforming mesh. Cases (b) and (d) represent nonaffine transformations.

The first step was to recognize the constraints imposed by the entropy inequality on the permeability tensor, which must be symmetric and positive semidefinite. This was followed by the recognition that the fourth-order tensor representing the rate of change of the permeability tensor with strain must satisfy minor symmetries but need not exhibit major symmetry.

Using the framework of representation theorems for symmetric tensor-valued functions, general expressions were provided for orthotropic, isotropic, and transversely isotropic permeability tensors. These tensor functions are dependent on scalar functions of the strain invariants, $k_i(I_j)=J^{-1}K_i(I_j)$, which, when positive for arbitrary strains, enforce positive-definiteness of the permeability tensor.

These general formulations demonstrated that, in general, strain induces anisotropy in the permeability tensor under finite deformation, even if the permeability tensor was isotropic in the reference configuration. Thus, the most commonly used permeability constitutive relation under finite deformation in soft tissue mechanics, Eq. (35) [14,24,29–37], was shown to represent only a special case, where permeability in the spatial frame remains isotropic regardless of the state of strain. Whether the permeability of soft tissues should obey Eq. (35) or a more general form that results in strain-induced anisotropy, such as Eq. (44), can only be resolved experimentally. However, the canonical problem of 1D permeation, which is most commonly used to measure permeability under finite deformation, cannot discriminate uniquely between these two forms of constitutive relations and is thus unable to resolve this question. The alternative canonical problem of permeation under torsion, described above, may be able to resolve this question if the flow of interstitial fluid can be visualized experimentally and shown to follow either a rectilinear or helical pattern (Fig. 1). In particular, an examination of whether or not the fluid flux pattern follows the deformation according to an affine transformation (Fig. 2) may determine the relative contribution of the various terms appearing in the general expression for the permeability tensor, such as in the example given in Eq. (44).

Depending on the material symmetry, the scalar functions $k_i(I_j)$ may depend on a narrower or broader set of strain invariants I_j . In the prior literature, only a dependence on $J=\sqrt{I_3}=\det \mathbf{F}$ has been proposed, as in Eq. (36), motivated in part by the pore closure constraint of Eq. (9). A dependence on other strain invariants would need to be motivated from experimental findings, using testing configurations that can discriminate among the contributions of the various strain invariants. The conceptualization and implementation of such testing configurations may be the subject of future studies.

The orthotropic, transversely isotropic, and isotropic strain-dependent formulations presented above model transport properties that exhibit those material symmetries in the reference con-

figuration but exhibit further anisotropy with applied finite strain. For example, the transversely isotropic model may be applicable to fibrous tissues with a single preferred fiber orientation, such as tendons and some ligaments, or for membranes or sheets with random in-plane fiber orientation. Even if these tissues undergo small strains under normal function, they may still experience large rotations (such as gliding tendons that wrap around bones, ligaments that span across flexing diarthrodial joints, or heart valve leaflets that undergo large bending). The preferred material directions for transport should conform to these large motions, as demonstrated in these formulations.

As a final note, though the presentation of this study focused on the hydraulic permeability tensor, every concept presented here is equally applicable to the diffusivity tensor for solute transport in deformable porous media. Diffusivity is a measure of frictional interactions between the solute and the interstitial solvent and porous solid matrix, just like permeability is a measure of frictional interactions between the interstitial solvent and porous solid. The diffusivity tensor must be symmetric and positive semidefinite according to the same thermodynamic principles leading to these constraints for the permeability tensor (Appendix). Consequently, the same method of formulating constitutive relations based on representation theorems of tensor-valued functions applies.

In summary, this study provided guidelines and formulations for anisotropic strain-dependent transport properties in porous-deformable media undergoing large deformations, such as biological tissues and cells. The general formulation accounted for the strain-induced anisotropy. Examples of constitutive relations for isotropic and orthotropic materials were illustrated, although the general framework presented here provides the foundation for any choice of constitutive relations.

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Appendix

In a mixture of a solid and a fluid, when the effects of viscosity are neglected relative to the effects of frictional interactions between the solid and fluid, under quasistatic conditions and in the absence of external body forces, the momentum equation for the fluid is given by

$$-\varphi^f \text{grad } p + \hat{\mathbf{p}}_d^f = \mathbf{0} \quad (\text{A1})$$

where $\hat{\mathbf{p}}_d^f$ is the dissipative part of the internal momentum to the fluid due to frictional interactions with the solid matrix [38–40]. A diffusive drag force $\hat{\mathbf{p}}_d^s$ also acts on the solid matrix. There are two constraints that need to be satisfied by $\hat{\mathbf{p}}_d^\alpha$ ($\alpha=s,f$). From the rule of mixtures, which states that a heterogeneous mixture must obey the ordinary equations of a continuum [41], we must have

$$\sum_{\alpha} \hat{\mathbf{p}}_d^{\alpha} = \mathbf{0} \quad (\text{A2})$$

in the absence of chemical reactions. Furthermore, from the entropy inequality, $\hat{\mathbf{p}}_d^{\alpha}$ must satisfy

$$\sum_{\alpha} \hat{\mathbf{p}}_d^{\alpha} \cdot \mathbf{v}^{\alpha} \leq 0 \quad (\text{A3})$$

where \mathbf{v}^{α} is the velocity of constituent α .

In the absence of chemical reactions, the general form for the diffusive drag that can satisfy Eq. (A3) must be

$$\hat{\mathbf{p}}_d^\alpha = \sum_{\beta} \mathbf{f}^{\alpha\beta} \cdot (\mathbf{v}^\beta - \mathbf{v}^\alpha) \quad (\text{A4})$$

where $\mathbf{f}^{\alpha\beta}$ ($\alpha \neq \beta$) is the diffusive drag tensor between constituents α and β . In general, $\mathbf{f}^{\alpha\beta}$ may be a function of the strain, temperature, magnitude $|\mathbf{v}^\alpha - \mathbf{v}^\beta|$ of the relative velocities, and fluid content. Equation (A2) can be satisfied for arbitrary velocities and mixtures of arbitrary pairs of constituents, if and only if

$$\mathbf{f}^{\alpha\beta} = \mathbf{f}^{\beta\alpha} \quad (\text{A5})$$

Substituting Eq. (A4) into Eq. (A3), recognizing that the dummy indices on the resulting double summation can be interchanged, and making use of Eq. (A5) yield the quadratic form

$$\sum_{\alpha} \sum_{\beta} (\mathbf{v}^\beta - \mathbf{v}^\alpha) \cdot \mathbf{f}^{\alpha\beta} \cdot (\mathbf{v}^\beta - \mathbf{v}^\alpha) \geq 0 \quad (\text{A6})$$

Therefore, the entropy inequality implies that the tensor $\mathbf{f}^{\alpha\beta}$ must be positive semidefinite (its eigenvalues must be non-negative [17]). An interesting property of the quadratic form of Eq. (A6) is that it places a constraint only on the symmetric part of $\mathbf{f}^{\alpha\beta}$. This is easily shown by splitting $\mathbf{f}^{\alpha\beta}$ into the sum of its symmetric and antisymmetric parts, and recognizing that the quadratic form of the latter reduces to zero. It follows that this antisymmetric part has no influence on the energy dissipated by this frictional interaction, represented by the term on the left-hand-side of Eq. (A6); consequently, it is *not* observable [42]. Based on this argument, $\mathbf{f}^{\alpha\beta}$ is taken to be symmetric, $(\mathbf{f}^{\alpha\beta})^T = \mathbf{f}^{\alpha\beta}$, since it can never be shown to be otherwise.

In the case of a solid-fluid mixture, the diffusive drag tensor \mathbf{f}^{fs} may be related to the spatial hydraulic permeability tensor \mathbf{k} via $\mathbf{f}^{fs} = (\varphi^f)^2 \mathbf{k}^{-1}$, so that $\hat{\mathbf{p}}_d^f = -\varphi^f \mathbf{k}^{-1} \cdot \mathbf{w}$, where $\mathbf{w} = \varphi^f (\mathbf{v}^f - \mathbf{v}^s)$ is the flux of interstitial fluid relative to the solid. Substituting this relation for $\hat{\mathbf{p}}_d^f$ into Eq. (A1) produces Darcy's law, as presented in Eq. (1). A symmetric, positive semidefinite \mathbf{f}^{fs} implies the same properties for \mathbf{k} .

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